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INTUITIONISTIC FUZZY SUPPLEMENT AND INTUITIONISTIC FUZZY COCLOSED SUBMODULES

P.K.Sharma[1], Gagandeep Kaur[2], Devinder Singh Pathania[3]

Abstract- In this paper, we introduce the concept of intuitionistic fuzzy supplement submodules of a module. We attempt to investigate various properties of such submodules. Also we define an intuitionistic fuzzy coclosed submodules and study the relationship with intuitionistic fuzzy supplement submodules.
Keywords – Intuitionistic fuzzy submodule, Intuitionistic fuzzy supplement submodule, Intuitionistic fuzzy coclosed submodule.

1. INTRODUCTION

After the introduction of intuitionistic fuzzy sets by Atanassov [1], [2] a number of applications of this fundamental concept have come up. Biswas was the first one to introduce the intuitionistic fuzzification of the algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Hur and others [10] applied this concept to rings and introduce the notion of intuitionistic fuzzy subrings and ideals of a ring. Davvaz et al. in [7] introduced the notion of intuitionistic fuzzy submodules of a module. Consequently, intuitionistic fuzzy module generated by intuitionistic fuzzy sets [13], intuitionistic fuzzy quotient modules [12], [14], residual quotient and annihilator of intuitionistic fuzzy sets of ring and module [18] were investigated. Basnet [3] defined intuitionistic fuzzy essential submodules and intuitionistic fuzzy divisible and pure submodules in [8] and investigated various characteristics of such submodule. Authors in [17] defined intuitionistic fuzzy small submodules and that of intuitionistic fuzzy cosmall submodule in [19].

In this paper we give the definition of intuitionistic fuzzy supplement submodules and intuitionistic fuzzy coclosed submodules of an intuitionistic fuzzy module. We study some of its properties and their relationship.

2. PRELIMINARIES

Throughout this section, R is a commutative ring with unity 1; $1 \neq 0$; M is a unitary R-module and θ is the zero element of M. The class of intuitionistic fuzzy subsets of X is denoted by IFS(X).

Definition (2.1). Let $K \leq M$. Then K is said to be a small submodule of M, if for all submodules L of M, $K + L = M$ implies that $L = M$. It is indicated by the notation $K << M$.

Definition (2.2). Let $K \leq M$. If $K \leq M$, then we say M is a small cover of M/K.

Definition (2.3). Let $N, L \leq M$. Then we say that N lies above K or K is a coessential submodule of N, if M/K is a small cover of M/N, that is; $N/K << M/K$.

Definition (2.4).Let $N \leq M$. N is called coclosed in M if and only if N has no proper coessential submodule, that is $\exists$ K, $N/K \not\ll M/K$.

Definition (2.5). Let $N, L \leq M$. Then we call N a supplement of L, if N is minimal with respect to $N + L = M$. Equivalently, N is called a supplement of L if and only if $N + L = M$ and $N \cap L << N$.

Definition (2.6). Let M be an R-module. If every submodule of M has a supplement, then M is called a supplemented module.

Definition (2.7). Let $N, L \leq M$. Then we call N a weak supplement of L if and only if $N + L = M$ and $N \cap L << M$.

Definition (2.8). Let M be an R-module. If every submodule of M has a weak supplement, then M is called a weakly supplemented module.

Definition (2.9). If for every $K \leq M$ with $K + N = M$, there is a supplement L of K such that $L \subseteq N$, then it is said that K has ample supplements in M.

Definition (2.10). If every submodule of M has ample supplements in M, then M is called amply supplemented. Every amply supplemented module is supplemented.

Definition (2.11). ([1]) Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X\}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

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[1] Post-Graduate Department of Mathematics and Communication Engineering, D.A.V. College, Jalandhar, Punjab, India
[2] Research Scholar, IKG PT University, Jalandhar, Punjab, India
[3] Department of Applied Sciences, Guru Nanak Dev Engineering College, Ludhiana, Punjab, India

Remark (2.12).
(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., when $\nu_A(x) = 1-\mu_A(x) = \mu_A^c(x)$: Then A is called a fuzzy set.
(ii) We denote the IFS $A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition (2.13) ([1], [2]) Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be IFSs of X. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x), \forall x \in X$;

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

$(iii) A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$;

$(iv) A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >: x \in X\}$;

$(v) A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >: x \in X\}$.

Definition (2.14) ([7], [11], [16]) Let M be a module over a ring R. An IFS $A=(\mu_A,\nu_A)$ of M is called an intuitionistic fuzzy submodule (IFSM) if

$(i)\mu_A(\theta)=1,\ \nu_A(\theta)=0,$ where θ is the zero element of M;

$(ii)\mu_A(x+y) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \le \nu_A(x) \vee \nu_A(y)$;

(iii) $\mu_A(rx) \ge \mu_A(x)$ and $\nu_A(rx) \le \nu_A(x), \forall\ x, y \in M, r \in R.$

Condition (i) and (ii) can be combined to a single condition

$\mu_A(rx+sy) \ge \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(rx+sy) \le \nu_A(x) \vee \nu_A(y) \forall\ x, y \in M, r, s \in R.$

The set of intuitionistic fuzzy submodules of R-module M is denoted by IFM(M).

Definition (2.15). We define two IFSs $\chi_{\{\theta\}} = (\mu_{\chi_{\{\theta\}}}, \nu_{\chi_{\{\theta\}}})$ and $\chi_M = (\mu_{\chi_M}, \nu_{\chi_M})$ of R-module M.

$$\mu_{\chi_{\{\theta\}}}(x) = \begin{cases} 1 & if\ x=\theta \\ 0 & if\ x\neq\theta \end{cases};\ \nu_{\chi_{\{\theta\}}}(x) = \begin{cases} 0 & if\ x=\theta \\ 1 & if\ x\neq\theta \end{cases},\ \text{and}\ \mu_{\chi_M}(x)=1;\ \nu_{\chi_M}(x)=0, \forall\ x \in M.$$

Then it can be easily verified that $\chi_{\{\theta\}}, \chi_M \in$IFM(M). These are called trivial IFSMs of module M. Any IFSM of module M other than these is called proper IFSM.

Definition (2.16). [17] Let $A=(\mu_A,\nu_A)$ be an IFS of X, then support of A is defined by A^* and is defined as

$A^* = \{x \in X :\ \mu_A(x) > 0$ and $\nu_A(x) < 1\}$ and we denote the set $A_* = \{x \in X :\ \mu_A(x) = 1$ and $\nu_A(x) = 0\}$.

By ([17], Proposition (2.16)) if A is an IFSM of M, then A_* is a submodule of M.

Definition (2.17). ([17]) Let A, B $\in$ IFM(M) be such that $A \subseteq B$. Then the quotient of B with respect to A is an IFSM of M/A^*, denoted by B/A, and is defined as $B/A(x+A^*) = (\mu_{B/A}(x+A^*),\ \nu_{B/A}(x+A^*))$, where
$\mu_{B/A}(x+A^*) = Sup\{\mu_B(x+y) : y \in A^*\}$ and $\nu_{B/A}(x+A^*) = Inf\{\nu_B(x+y) : y \in A^*\}$, where $x \in B^*$.

Definition (2.18) ([17]) Let M be an R-module and A $\in$ IFM(M). Then A is said to be an intuitionistic fuzzy small (superfluous) submodule (IFSSM) of M, if for any B $\in$IFM(M), $A+B = \chi_M \Rightarrow B = \chi_M$. It is denoted by the notation $A <<_{IF} M$ or $A <<_{IF} \chi_M$.

It is obvious that $\chi_{\{\theta\}}$ is always an IFSSM of M.

Definition (2.19). ([17], [19]) Let A and B be any two intuitionistic fuzzy submodule of M such that $A \subseteq B$, then A is called an intuitionistic fuzzy submodule of B. A is called an intuitionistic fuzzy small submodule in B, denoted by $A <<_{IF} B$ or $A <<_{IF} B^*$.

Proposition (2.20). ([17]) Let $N \le M$. Then $N << M$ if and only if $\chi_N <<_{IF} M$.

Proposition (2.21). ([17]) Let $A \in IFM(M)$. Then $A \ll_{IF} M$ if and only if $A_* \ll M$.
Proposition (2.22). ([19]) Let $A, B \in IFM(M)$. Then $A \ll_{IF} B$ if and only if $A_* \ll B_*$.

3. INTUITIONISTIC FUZZY SUPPLEMENT SUBMODULES AND INTUITIONISTIC FUZZY COCLOSED SUBMODULES

Definition (3.1). Let $B \in IFM(M)$. B is called an intuitionistic fuzzy coclosed in M if and only if B has no proper intuitionistic fuzzy coessential submodule, that is $\nexists A \in IFM(M)$ such that $B/A \ll_{IF} \chi_M /A$.
Proposition (3.2). If B is an intuitionistic fuzzy coclosed in M, then B_* is coclosed in M.
Proof. Let B be an intuitionistic fuzzy coclosed in M. Assume K be a submodule of B_* such that $B_* /K \ll M/K$. So $B/\chi_K \ll_{IF} \chi_M /\chi_K$ and since B is an intuitionistic fuzzy coclosed in M, thus $\chi_K = B$ and therefore $K = B_*$. Hence B_* is coclosed in M.
Remark (3.3). Inverse of previous proposition is not essentially true.
Lemma (3.4). Let $A, B \in IFM(M)$ such that $A \subseteq B$. If B is an intuitionistic fuzzy coclosed in M and $A \ll_{IF} M$, then $A \ll_{IF} B$.
Proof. Assume $C \subseteq B$ and $A + C = B$. Let $D \in IFM(M)$ and $C \subseteq D$ such that $\chi_M /C = B/C + D/C$. Thus $\chi_M = B + D = A + C + D$ and since $A \ll_{IF} M$, therefore $\chi_M = C + D$. Since $C \subseteq D$, thus $\chi_M = D$. Hence $B/C \ll_{IF} \chi_M /C$ and since B is an intuitionistic fuzzy coclosed in M, thus B has no proper intuitionistic fuzzy coessential submodule. So $C = B$ and hence $A \ll_{IF} B$.
Definition (3.5). Let $A, B \in IFM(M)$ such that $A \subseteq B$. A is called an intuitionistic fuzzy coclosure of B in M, if A is an intuitionistic fuzzy coessential submodule of B and A has no proper intuitionistic fuzzy coessential submodule.
Lemma (3.6). Let $A, B \in IFM(M)$ such that $A \subseteq B$. If $B = A + C$ and $C \ll_{IF} M$, then A is an intuitionistic fuzzy coessential submodule of B in M.
Proof. Suppose $D \in IFM(M)$ such that $\chi_M = B + D$. Thus $\chi_M = A + C + D$ and since $C \ll_{IF} M$, hence $\chi_M = A + D$. By proposition 2.23, A is an intuitionistic fuzzy coessential submodule of B.
Definition (3.7) Let $A, B \in IFM(M)$. Then we call A is an intuitionistic fuzzy supplement of B , if A is minimal with respect to $A + B = \chi_M$. Equivalently, A is called an intuitionistic fuzzy supplement of B if and only if $A + B = \chi_M$ and $A \cap B \ll_{IF} A$.
An intuitionistic fuzzy submodule A of M is called an intuitionistic fuzzy supplement, if there is an intuitionistic fuzzy submodule B of M such that A is an intuitionistic fuzzy supplement of B.
Definition (3.8). Let $A, B \in IFM(M)$. A is called a weak intuitionistic fuzzy supplement of B if and only if $A + B = \chi_M$ and $A \cap B \ll_{IF} M$.
An intuitionistic fuzzy submodule A of M is called a weak intuitionistic fuzzy supplement, if there is an intuitionistic fuzzy submodule B of M such that A is a weak intuitionistic fuzzy supplement of B.
Clearly any intuitionistic fuzzy supplement submodule is a weak intuitionistic fuzzy supplement submodule.
Proposition (3.9). Let $A, B \in IFM(M)$. If A and B have finite images, then A is an intuitionistic fuzzy supplement of B if and only if A_* is a supplement of B_* .
Proof. Let A be an intuitionistic fuzzy supplement of B. So $A + B = \chi_M$ and $A \cap B \ll_{IF} A$. Thus $(A + B)_* = M$ and $(A \cap B)_* \ll A_*$. By proposition (2.18), $A_* + B_* = M$ and $A_* \cap B_* \ll A_*$. Hence A is a supplement of B .
Conversely, assume A_* is a supplement of B_* . So $A_* + B_* = M$ and $A_* \cap B_* \ll A_*$. By proposition 2.18, $(A + B)_* = M$ and $(A \cap B)_* \ll A_*$. Thus $A + B = \chi_M$ and $A \cap B \ll_{IF} A$. Hence A be an intuitionistic fuzzy supplement of B.
Corollary (3.10). Let $A, B \in IFM(M)$. Then A is a weak intuitionistic fuzzy supplement of B if and only if A_* is a weak supplement of B_* .
Definition (3.11). An R-module M (or χ_M) is said to be an intuitionistic fuzzy hollow module, if every proper intuitionistic fuzzy submodule of M is an intuitionistic fuzzy small submodule in M.
Remark (3.12).
(a) Let A be an intuitionistic fuzzy hollow submodule of an R-module M. If A is not intuitionistic fuzzy small in M, then there exists $C \in IFM(M)$ such that $C \neq \chi_M$ and $A + C = M$. Since A is an intuitionistic fuzzy hollow, $A \cap C \ll_{IF} A$. Thus A is an intuitionistic fuzzy supplement in M.
(b) Let $A, B \in IFM(M)$, $A \subseteq B$. By (2.23), B lies above A if and only if $B + C = \chi_M$ implies $A + C = \chi_M$ for all $C \in IFM(M)$. If B is minimal with respect $B + C = \chi_M$ for some B, then there cannot be an intuitionistic fuzzy submodule A of B such that B lies above A. Thus B is an intuitionistic fuzzy coclosed.
Proposition (3.13). Let $A \in IFM(M)$. Consider the following statement:
(i) A is an intuitionistic fuzzy supplement in M;
(ii) A is an intuitionistic fuzzy coclosed in M;
(iii) For all $B \subseteq A$, $B \ll_{IF} M$ implies $B \ll_{IF} A$.

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds and if A is weak intuitionistic fuzzy supplement in M then (iii) $\Rightarrow$ (i) hold.

Proof. (i) $\Rightarrow$ (ii) Assume that A is an intuitionistic fuzzy supplement of B $\in$ IFM(M). For all intuitionistic fuzzy submodules C $\subseteq$ A such that A lies above C, we have that A + B = χ_M implies C + B = χ_M . By the minimality of A with respect to this property, we get A = C. Hence A is an intuitionistic fuzzy coclosed.
(ii) $\Rightarrow$ (iii) Let B $<<_{IF}$ M and B $\subseteq$ A. Assume A = B + C for C $\subseteq$ A. Then for every D $\in$ IFM(M) with A + D = χ_M , we get B + C + D = χ_M . Since B $<<_{IF}$ M, so C + D = χ_M . Thus C is an intuitionistic fuzzy coessential of A. By the coclosure of A, we get C = A. Thus B $<<_{IF}$ A.

(iii) $\Rightarrow$ (i) Assume A is a weak intuitionistic fuzzy supplement of C $\in$ IFM(M), so A $\cap$ C $<<_{IF}$ M. By assumption, A $\cap$ C $<<_{IF}$ A. Thus A is a supplement of C in M.

Corollary (3.14). Any intuitionistic fuzzy direct summand of M is an intuitionistic fuzzy coclosed in M.
Proposition (3.15). Let A, B $\in$ IFM(M) such that A $\subseteq$ B
(i) If B is an intuitionistic fuzzy coclosed in M, then B/A is an intuitionistic fuzzy coclosed in χ_M =A.
(ii) Assume B is an intuitionistic fuzzy supplement in M. Then A is an intuitionistic fuzzy coclosed in B if and only if A is an intuitionistic fuzzy coclosed in M.
Proof. (i) Since B is an intuitionistic fuzzy coclosed in M, for every proper intuitionistic fuzzy submodule D/A of B/A, (B/A)/(D/A) $\cong$ B/D is not intuitionistic fuzzy small in (χ_M /A)/(D/A) $\cong$ χ_M /D (since B/D is not small in χ_M /D). So B/A is an intuitionistic fuzzy coclosed in χ_M /A.

(ii)Let B be intuitionistic fuzzy supplement of D $\subseteq$ χ_M . Assume A is an intuitionistic fuzzy coclosed in M. Since whenever A/E $<<_{IF}$ B/E, we get A/E $<<_{IF}$ χ_M /E as B/E $\subseteq$ χ_M /E, so A is an intuitionistic fuzzy coclosed in B.

Conversely, assume that A is an intuitionistic fuzzy coclosed in B and that A lies above a proper intuitionistic fuzzy submodule F $\subseteq$ A in M. Since A is an intuitionistic fuzzy coclosed in B and A does not lie above in F , hence there exists a proper intuitionistic fuzzy submodule G of B containing such that A/F + G/F = B/F holds. Hence χ_M = B + D = A + G + D implies χ_M = F + G + D = F + D (since A lies above in M). But since B is an intuitionistic fuzzy supplement of B in M, we get G = B, a contradiction to G being proper intuitionistic fuzzy submodule of B. Hence A is an intuitionistic fuzzy coclosed in M.

Definition (3.16). An R-module M (or χ_M) is called an intuitionistic fuzzy supplemented, if every intuitionistic fuzzy submodule has an intuitionistic fuzzy supplement in M.
Definition (3.17). An R-module M (or χ_M) is called weakly intuitionistic fuzzy supplemented, if every intuitionistic fuzzy submodule has a weak intuitionistic fuzzy supplement in M.
Lemma (3.18). Any intuitionistic fuzzy supplemented module is weakly intuitionistic fuzzy supplemented.
Proof. Suppose M be an intuitionistic fuzzy supplemented and A $\in$ IFM(M). So there is B $\in$ IFM(M) such that A+B = χ_M and A $\cap$B $<<_{IF}$ B. Thus A $\cap$ B$<<_{IF}$ M, so B is a weak intuitionistic fuzzy supplement of A. Hence M is weakly intuitionistic fuzzy supplemented.
Proposition (3.19). Let intuitionistic fuzzy submodules of R-module M have finite images. Then R-module M is an intuitionistic fuzzy supplemented if and only if M is supplemented module.
Proof. First let M be an intuitionistic fuzzy supplemented. Let N $\leq$ M and A be the characteristic function on N, i.e., A = χ_N . So A $\in$ IFM(M) and A_* = N. Since M is intuitionistic fuzzy supplemented, so there is B $\in$ IFM(M) such that A + B = χ_M and A $\cap$ B $<<_{IF}$ M. Hence $(A + B)_*$ = M and $(A \cap B)_*$ << M. By proposition (2.18) , $A_* + B_*$ = M and $A_* \cap B_*$ << M where B_* is submodule of M. Hence B_* is a supplement of N. This implies M is supplemented module.
Conversely, assume M is supplemented module. Let A $\in$ IFM(M), so A_* << M. Since M is supplemented, thus there is K $\leq$ M such that A_* + K = M and $A_* \cap$ K << M. Suppose B is the characteristic function on K, i.e., B = χ_K so B $\in$ IFM(M) and B_* = K. Therefore $A_* + B_*$ = M and $A_* \cap B_*$ << M. So by proposition (2.18) , $(A + B)_*$ = M and $(A \cap B)_*$ << M. Thus A + B = χ_M and A $\cap$ B $<<_{IF}$ M. Hence B is an intuitionistic fuzzy supplement of A and so M is an intuitionistic fuzzy supplemented module.
Corollary (3.20). An R-module M is weakly intuitionistic fuzzy supplemented if and only if M is weakly supplemented module.
Example (3.21). Semi-simple module is an intuitionistic fuzzy supplemented.
Example (3.22). The Z-module Q is weakly intuitionistic fuzzy supplemented.
Proposition (3.23). Assume that M is weakly intuitionistic fuzzy supplemented and A, B $\in$ IFM(M) such that χ_M = A + B. Then A has a weak intuitionistic fuzzy supplement as C in M such that C $\subseteq$ B.
Proof. Let χ_M = A + B. Since M is weakly intuitionistic fuzzy supplemented, so there is D $\in$ IFM(M) such that χ_M = (A $\cap$ B) + D and (A $\cap$ B) $\cap$ D $<<_{IF}$ M. Thus B = χ_M $\cap$ B = ((A $\cap$ B) +D) $\cap$ B = (A $\cap$ B) + (D $\cap$ B). So χ_M = A+ (A $\cap$ B) + (D $\cap$ B) and A $\cap$ (D $\cap$ B) $<<_{IF}$ M. Since A $\cap$ B $\subseteq$ A, so χ_M = A + (D $\cap$ B). Hence (D $\cap$B) = C (say) is a weak intuitionistic fuzzy

supplement of A such that $C = (D \cap B) \subseteq B$.

Proposition (3.24). Let M be weakly intuitionistic fuzzy supplemented and $A \in IFM(M)$. Then the following conditions are equivalent:
(i) A is an intuitionistic fuzzy supplement;
(ii)A is an intuitionistic fuzzy coclosed.

Proof. (i) $\Rightarrow$ (ii) By proposition (3.13).
(ii) $\Rightarrow$ (i) Let A be an intuitionistic fuzzy coclosed in M. Since M is weakly intuitionistic fuzzy supplement, so A has a weak supplement as B such that $\chi_M = A + B$ and $A \cap B <<_{IF} M$. By lemma (3.4), $A \cap B <<_{IF} A$. Thus A is an intuitionistic fuzzy supplement of B in M. Therefore A is an intuitionistic fuzzy supplement submodule in M.

4. AMPLY INTUITIONISTIC FUZZY SUPPLEMENTED MODULES

Definition (4.1). If for every $B \in IFM(M)$ with $A + B = \chi_M$ there is a supplement C of A such that $C \subseteq B$, then it is said that A has ample intuitionistic fuzzy supplement in M.
Definition (4.2). If every intuitionistic fuzzy submodule of M has a ample intuitionistic fuzzy supplement in M, then M (or χ_M) is called amply intuitionistic fuzzy supplemented.
Every amply intuitionistic fuzzy supplemented module is an intuitionistic fuzzy supplemented.
Proposition (4.3). Let every intuitionistic fuzzy submodules of R-module M have finite images. If R-module M is amply intuitionistic fuzzy supplemented, then M is amply supplemented.
Proof. Let M be amply intuitionistic fuzzy supplemented, let $N \leq M$ with $N + K = M$. Suppose A and B are the characteristic function on N and K. So $A, B \in IFM(M)$ and $A_* = N$, $B_* = K$. Thus $A_* + B_* = M$. By proposition (2.18), $A_* + B_* = (A + B)_* = M$, so $A + B = \chi_M$. Since M is amply intuitionistic fuzzy supplemented, so there is an intuitionistic fuzzy supplement C of A such that $C \subseteq B$. So by proposition (3.9), C_* is a supplement of A_* such that $C_* \subseteq B_*$. Since $C \in IFM(M)$, so $C_* = L \leq M$ and hence L is a supplement of N such that $L \subseteq K$. Therefore M is amply supplemented.
Proposition (4.4). An R-module M is amply intuitionistic fuzzy supplemented if and only if M is weakly intuitionistic fuzzy supplemented and any intuitionistic fuzzy submodule of M has an intuitionistic fuzzy coclosure in M.
Proof. Let M be amply intuitionistic fuzzy supplemented, so M is weakly intuitionistic fuzzy supplemented. Let $A \in IFM(M)$, thus A has an intuitionistic fuzzy supplement as B. So $A + B = \chi_M$ and $A \cap B <<_{IF} B$, this implies $A \cap B <<_{IF} M$. Since M is amply intuitionistic fuzzy supplemented, so there is an intuitionistic fuzzy supplement C of B such that $C \subseteq A$. Thus $C + B = \chi_M$ and $C \cap B <<_{IF} C$. We will prove C is an intuitionistic fuzzy coclosure of A in M. C is intuitionistic fuzzy coclosed in M because C is an intuitionistic fuzzy supplement submodule in M. Since $A = A \cap \chi_M = A \cap (C + B) = (A \cap C) + (A \cap B) = C + (A \cap B)$ and $A \cap B <<_{IF} M$, by lemma (3.18), C is an intuitionistic fuzzy coessential submodule of A in M. Therefore C is intuitionistic fuzzy coclosure of A in M.
Conversely, let M is weakly intuitionistic fuzzy supplemented and any intuitionistic fuzzy submodule of M has an intuitionistic fuzzy coclosure in M. Let $A, B \in IFM(M)$ and $\chi_M = A + B$. Since M is weakly intuitionistic fuzzy supplemented and by proposition (2.18), there is a weak intuitionistic fuzzy supplement C of A such that C B. Thus $\chi_M = A + C$ and $A \cap C <<_{IF} M$. Assume D is an intuitionistic fuzzy coclosure of C in M. So by proposition (2.23), $\chi_M = A + D$. Since D is an intuitionistic fuzzy coclosed in M and $D \cap A <<_{IF} M$, so by lemma (3.4), $D \cap A <<_{IF} D$. This implies D is an intuitionistic fuzzy supplement of A in M such that $D \subseteq B$. Hence M is amply intuitionistic fuzzy supplemented.
Proposition (4.5). Let M is amply intuitionistic fuzzy supplemented module. Then,
(i) Any intuitionistic fuzzy supplement of an intuitionistic fuzzy submodule of M is a amply intuitionistic fuzzy supplemented module.
(ii) Any intuitionistic fuzzy direct summand of M is a amply intuitionistic fuzzy supplemented module.

Proof. (i) Let $A \in IFM(M)$ and B is an intuitionistic fuzzy supplement of A such that for intuitionistic fuzzy submodules C and D of B, $B = D + C$. Since B is an intuitionistic fuzzy supplement of A, so $\chi_M = A + B = A + D + C$ and since M is amply intuitionistic fuzzy supplemented, thus there is an fuzzy supplement E of A + D such that $E \subseteq C$. Also, $D \subseteq A + C$, so $D \cap E \subseteq (A + D) \cap E$. Since E is an intuitionistic fuzzy supplement of D + A, thus $D \cap E \subseteq (A + D) \cap E <<_{IF} E$. Hence $\chi_M = A + D + E$ and $D \cap E <<_{IF} E$. Now, by the minimality of B with respect to this property, we get $B \subseteq D + E$. Since $D \subseteq B$ and $E \subseteq C \subseteq B$, so $D + E \subseteq B$. Hence $B = D + E$, this implies is an intuitionistic fuzzy supplement of D in B and so B is amply intuitionistic fuzzy supplemented.

(ii) Since any intuitionistic fuzzy direct summand is an intuitionistic fuzzy supplement, so by (i), any intuitionistic fuzzy direct summand is a amply intuitionistic fuzzy supplemented module.

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